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Introduction

The purpose of this note is to give a survey of M. Borovoi's work on Hasse principle and approximation theorems for homogeneous spaces [B2] $\sim$ [B6]. An ingenious tool to study such arithmetic is ***abelianization of Galois cohomology*** of affine algebraic groups [B1].

For principal homogeneous spaces, this theme is, in the algebraic aspect, the theory of Galois cohomology of affine algebraic groups, which has been developed by Borel, Serre, Tate, Voskresenskii, Kneser, Harder, Platonov, Chernousov etc ([BS], [T], [V], [Kn1-4], [Har1,2], [P], [Ch] etc). A basic reference is the book by Platonov and Rapinchuk [PR]. Sansuc [Sa] discussed these stuffs in light of the brilliant idea using the Brauer-Grothendieck group, introduced by Manin [Ma1,2] for Hasse principle and by Colliot-Thélène, Sansuc [CTS] for weak approximation. The problem here is the uniqueness of the Brauer-Manin obstruction.

On the other hand, Kottwitz [Ko1,2] showed some analogies for reductive groups to duality theorems for Galois cohomology of abelian algebraic groups. He described the results in terms of the Langlands dual group $\hat{G}$. Since the correspondence $G \to \hat{G}$ is not functorial in G, his duality theorems were stated in the category where the morphisms are normal. Borovoi [B1] reconstructed Kottwitz's results, based on the theory of non-abelian hypercohomology with coefficients in crossed modules, and described them in functorial manner using the algebraic fundamental groups. So, the results are naturally expected to be applied to some arithmetic questions on homogeneous spaces. Moreover, since the Picard group of an affine algebraic group is expressed by the algebraic fundamental group, we can describe these cohomological results in terms of the Brauer-Grothendieck group.

In this note, I tried to explain the technical background of abelian Galois cohomology rather precisely, and roughly touched the Brauer-Manin business which I hope to write up precisely in the future.

It is a pleasure to thank Takao Watanabe for useful conversations on this topic. I am also grateful to Prof. M. Borovoi for answering my questions.

Notations. Let k be a field of characteristic zero. We denote by $\overline{k}$ an fixed algebraic closure and by Γ_k the Galois group of $\overline{k}/k$. For a connected affine k-group G, G^u is the unipotent radical, $G^{red} = G/G^u$, G^{ss} is the derived group of G^{red}, G^{sc} is the simply connected covering of G^{ss}, $G^{ad} = G/Z$, Z is the center of G. When k is a number field, $\mathcal{V}_k = \mathcal{V}$ is the set of all places of k, $\mathcal{V}_f$ (resp. $\mathcal{V}_\infty$) is the set of all finite (resp. infinite) places of k. For $v \in \mathcal{V}$, k_v denotes the completion of k at v. For a finite subset S of $\mathcal{V}$, $k_S = \prod_{v\in S} k_v$ and $\mathbf{A}^S$ is the ring of adeles without S-components, the adele ring of k is $\mathbf{A} = k_S \times \mathbf{A}^S$. The cohomology class of a cocycle c is denoted by $Cl(c)$.

1. Quadrics

Let Q be a non-singular quadratic form in n variables over a number field k. Consider the affine quadric

$$X = \{x\,;\, Q(x) = a\} \quad (a \in k^\times)$$

Hasse Principle : $X(k) \neq \phi$ *or* $\prod_{v\in\mathcal{V}} X(k_v) = \phi$.

Weak approximation with respect to S***: For any finite set*** S ***of places of*** k***, the diagonal image of*** $X(k)$ ***in*** $X(k_S)$ ***is dense.***

Strong approximation with respect to S***: There is a finite set*** S ***of places of*** k***, the diagonal image of*** $X(k)$ ***in*** $X(\mathbf{A}^S)$ ***is dense.***

The Hasse principle for a quadric over any number field was proved by Hasse [Has]. The weak approximation property follows from that X is rational over k. On the other hand, the strong approximation property has the group theoretic and arithmetic nature. To see this, write $X = H\backslash G$ for $n > 2$, where G is the spinor group of Q. Then, as we shall see in 4.3, we can take, as S, one finite prime for $n > 4$, and one finite prime which is inert or ramified in K/k for $n = 3$ where K is a splitting field of the torus H.

With this classical example in mind, our interest is when these properties for a homogeneous space of an affine algebraic k-group, especially, of a semisimple, simplyconnected k-group, are satisfied. What can we say about the obstructions ?

2. Algebraic fundamental group and abelianization of Galois cohomology

We recall some ingredients in Galois cohomology of affine algebraic groups to study arithmetical properties of homogeneous spaces. A basic reference is Borovoi's article [B1].

2.1. Let G be a connected affine k-group, k being a field of characteristic zero. First assume that G is reductive. Let G^{ss} be the derived group of G and G^{sc} the simplyconnected covering of G^{ss} over k. Consider the composition

$$\rho:\ G^{sc} \longrightarrow G^{ss} \subset G.$$

Let T be a maximal torus of G defined over k and set $T^{sc} = \rho^{-1}(T)$. We then define

$$\pi_1(G) = X_*(T)/\rho_* X_*(T^{sc})$$

where X_* denotes the cocharacter group of a torus. It is easy to see that Γ_k-module $\pi_1(G)$ does not depend on the choice of T. For a connected k-group G, we set $\pi_1(G) = \pi_1(G/G^u)$ and call it ***the algebraic fundamental group of*** G. Then $\pi_1(\cdot)$ is an exact functor from the category of connected affine k-groups to Γ_k-modules generated finitely over $\mathbf{Z}$. We see that an inner twisting does not change π_1, and if $k \subset \mathbf{C}$, $\pi_1(G)$ is isomorphic to the topological fundamental group of the complex Lie group $G(\mathbf{C})$ as abelian groups. Let $Z(\hat{G})$ be the center of the connected Langlands group $\hat{G}$ of a connected reductive group G. Then, $\pi_1(G)$ and the character group $X^*(Z(\hat{G}))$ are isomorphic as Γ_k-modules.

Next, we define ***the abelian Galois cohomology*** of a connected reductive

k-group G by

$$H^i_{ab}(k,G) = \mathbf{H}^i(k, T^{sc} \to T) \quad (i \geq -1)$$

Here, $\mathbf{H}^i$ means the Galois hypercohomology of the complex

$$0 \to T^{sc} \to T \to 0$$

where T^{sc} is in degree -1 and T is in degree 0. Noting that $X_*(T^{sc}) \to X_*(T) \to \pi_1(G)$ is a short torsion free resolution of $\pi_1(G)$ and that $S(\overline{k}) = X_*(S) \otimes \overline{k}$ for a k-torus S, we see that $H^i_{ab}(k,G)$ depends only on Γ_k-module $\pi_1(G)$. For a connected k-group G, we set $H^i_{ab}(k,G) = H^i_{ab}(k,G/G^u)$. On the other hand, for a connected reductive k-group G, $\rho : G^{sc} \to G$ is a crossed module where the action of G on G^{sc} is given via $G \to G^{ad} = (G^{sc})^{ad} \subset \mathrm{Aut}(G^{sc})$. So, we can define, in terms of cocycles, the non-abelian hypercohomology

$$\mathbf{H}^i(k, G^{sc} \to G)$$

for $i = -1, 0, 1$ in functorial way. Using the morphism $(1 \to G) \to (G^{sc} \to G)$ and the quasi-isomorphism $(T^{sc} \to T) \to (G^{sc} \to G)$ of crossed modules, we define ***the abelianization maps***

$$ab^i : H^i(k,G) \to H^i_{ab}(k,G)$$

for $i = 0, 1$. For a connected k-group G, the abelianization maps are defined by the composition

$$H^i(k,G) \to H^i(k,G/G^u) \xrightarrow{ab^i} H^i_{ab}(k,G/G^u) = H^i_{ab}(k,G)$$

For the center Z of G and $Z^{sc} = \rho^{-1}(Z)$, we see that $(Z^{sc} \to Z) \to (T^{sc} \to T)$ is a quasi-isomorphism and so $H^i_{ab}(k,G) = \mathbf{H}^i(k, Z^{sc} \to Z)$.

Example. Let G be a semisimple k-group. Then, $\pi_1(G) = \mathrm{Ker}\rho(-1), H^i_{ab}(k,G) = H^{i+1}(k, \mathrm{Ker}\rho)$ and ab^i, $i = 0, 1$ are connecting homomorphism attached to the exact sequence $1 \to \mathrm{Ker}\rho \to G^{sc} \xrightarrow{\rho} G \to 1$.

2.2. We recall non-abelian H^2 and its abelianization. The references are [B5] and [Sp]. Let $\overline{G}$ be a connected affine $\overline{k}$-group. Let $\mathrm{SAut}(\overline{G})$ denote the group of semi algebraic automorphisms of $\overline{G}$ and set $\mathrm{SOut}(\overline{G}) = \mathrm{SAut}(\overline{G})/\mathrm{Int}(\overline{G})$, $\mathrm{Int}(\overline{G})$ is the group of inner automorphisms of $\overline{G}$. Let $L = (\overline{G}, \kappa)$ be a

k-kernel, where $\kappa : \Gamma_k \to \mathrm{SOut}(\overline{G})$ is a homomorphism satisfying certain conditions. A 2-cocycle with coefficient in L is a pair (f, u) of continuous maps

$$f : \Gamma_k \longrightarrow \mathrm{SAut}(\overline{G}), \quad f \mod \mathrm{Int}(\overline{G}) = \kappa,$$
$$u : \Gamma_k \times \Gamma_k \longrightarrow \overline{G}(\overline{k})$$

satisfying 2-cocycle conditions.
The Galois cohomology set $H^2(k, L)$ is defined to be the quotient of the set of all 2-cocycles under the action of the group of continuous maps $\Gamma_k \to \overline{G}(\overline{k})$. A neutral element of $H^2(k, L)$ is the class of a cocycle of the form $(f, 1)$. A typical example of k-kernel is given as follows. Let G be a k-group. Each σ induces a semialgebraic automorphism $f_G(\sigma) = \sigma_*$ of $\overline{G} = G \otimes \overline{k}$. Set $\kappa_G = f_G \mod \mathrm{Int}\overline{G}$. Then, $(\overline{G}, \kappa_G)$ is a k-kernel and the class of $(f_G, 1)$ determines the neutral element $n(G)$ in $H^2(k, \overline{G}, \kappa_G)$. In fact, it is shown that any neutral element in $H^2(k, L)$ is of the form $n(G)$ for some k-form G of $\overline{G}$.
Our aim is to construct an abelian group $H^2_{ab}(k, L)$ and a map $ab^2 : H^2(k, L) \to H^2_{ab}(k, G)$ having the property that $\eta \in H^2(k, L)$ is neutral if and only if $ab^2(\eta) = 0$. In fact, it is possible when k is a local or a number field.

Assume $\overline{G}$ is reductive and let $\rho : \overline{G}^{sc} \to \overline{G}^{ss} \subset \overline{G}$ be the composition map. Note that κ defines k-forms Z, Z^{ss} and Z^{sc} of the centers of $\overline{G}, \overline{G}^{ss}$ and $\overline{G}^{sc}$, respectively. We then define ***the second abelian Galois cohomology group*** by

$$H^2_{ab}(k, L) = \mathbf{H}^2(k, Z^{sc} \to Z)$$

where we regard $Z^{sc} \to Z$ as the complex of Γ_k-modules with degree -1 for Z^{sc} and 0 for Z.
To define ab^2, we need two facts. First, the abelian group $H^2(k, Z)$ acts on $H^2(k, L)$ by $Cl(\varphi) + Cl(f, u) = Cl(f, \varphi u)$. It is shown [Sp] that the action is simply transitive. The second one is Douai's theorem [D] which guarantees the existence of a neutral element in $H^2(k, L)$. So, for $x \in H^2(k, L)$ and a neutral element $\eta \in H^2(k, L)$, there is a unique element $z \in H^2(k, Z)$ such that $x = z + \eta$. The short exact sequence of complexes

$$1 \to (1 \to Z) \to (Z^{sc} \to Z) \to (Z^{sc} \to 1) \to 1$$

gives rise to the exact sequence

$$\cdots \to H^2(k, Z^{sc}) \xrightarrow{\rho_*} H^2(k, Z) \xrightarrow{i_*} H^2_{ab}(k, L) \to \cdots$$

We then define *the abelianization map*

$$ab^2 : H^2(k, L) \longrightarrow H^2_{ab}(k, L)$$

by $ab^2(x) = i_*(z)$. It is checked the definition does not depend on the choice of η. For a connected k-kernel L, $H^2_{ab}(k, L)$ and ab^2 are defined in the similar way as in the case H^1

Now, suppose k is a local or a number field. Assume that $ab^2(\eta) = 0$, $\eta \in H^2(k, L)$. Then, there is a k-form G of $\overline{G}$ so that $\eta - n(G) = \rho_*(\chi)$ for some $\chi \in H^2(k, Z^{sc})$. By Kneser and Harder, χ comes from $H^1(k, (G^{sc})^{ad}) = H^1(k, G^{ad})$ under the connecting map. It yields η is neutral.

Theorem 2.2.([B5]) *Let k be a local or a number field. Let $L = (\overline{G}, \kappa)$ be a connected k-kernel. Then, $\eta \in H^2(k, L)$ is neutral if and only if $ab^2(\eta) = 0$.*

3. Kottwitz-Borovoi duality and Hasse principle for H^2

3.1. Let G be a connected reductive k-group. By definition in 2.1,

$$H^i_{ab}(k, G) = \mathbf{H}^i(k, T^{sc} \to T) = \mathbf{H}^i(k, (X_*(T^{sc}) \to X_*(T)) \otimes \overline{k}^\times).$$

So, the computation of abelian Galois cohomology for a local or a number field k is essentially Tate-Nakayama theory. The surjectivity of ab^1 below is proved using fundamental tori and all main results of Galois cohomology of affine algebraic groups.

Theorem 3.1.1.([Ko2],[B1]) *Let G be a connected affine group over a local field k of characteristic zero. Then, the map ab^1 is surjective and functorial in G.*

If k is nonarchimedean, ab^1 is bijective and there is a canonical, functorial in G, isomorphism

$$H^1_{ab}(k, G) \simeq \pi_1(G)_{\Gamma_k, tors}$$

where the r.h.s is the torsion subgroup of the coinvariant quotient of $\pi_1(G)$ under Γ_k.

If $k = \mathbf{R}$, there is a canonical, functorial in G, isomorphism

$$H^1_{ab}(\mathbf{R}, G) \simeq H^1(\mathbf{R}, \pi_1(G)) = \hat{H}^{-1}(\mathbf{R}, \pi_1(G)).$$

When k is a number field, the 1st non-abelian Galois cohomology can be computed in terms of abelian and real cohomology.

Theorem 3.1.2.([B1]) *Let G be a connected affine group over a number field k. Then, we have a commutative diagram*

$$\begin{tikzcd} H^1(k,G) \arrow[r, "ab^1"] \arrow[d, "\mathrm{loc}_\infty"'] & H^1_{ab}(k,G) \arrow[d, "\mathrm{loc}_\infty"] \\ \prod_{v \in \mathcal{V}_\infty} H^1(k_v, G) \arrow[r, "\prod_\infty ab^1_v"] & \prod_{v \in \mathcal{V}_\infty} H^1_{ab}(k_v, G) \end{tikzcd}$$

where all the maps are surjective.

By Theorem 3.1.1, for each $v \in \mathcal{V}$, we have a canonical, functorial in G, map

$$\mu_v \ : \ H^1(k_v, G) \longrightarrow \pi_1(G)_{\Gamma_{k_v}, tors}$$

Composing it with the corestriction

$$\lambda_v \ : \ \pi_1(G)_{\Gamma_{k_v}, tors} \longrightarrow \pi_1(G)_{\Gamma_k, tors}$$

and taking the sum, we get

$$\mu \ : \ \oplus_{v \in \mathcal{V}} H^1(k_v, G) \longrightarrow \pi_1(G)_{\Gamma_k, tors}.$$

Let $loc \ : \ H^1(k, G) \longrightarrow \oplus_{v \in \mathcal{V}} H^1(k_v, G)$ be the localization map. Then we have the following global duality.

Theorem 3.1.3. ([Ko2],[B1])

$$H^1(k, G) \xrightarrow{loc} \oplus_{v \in \mathcal{V}} H^1(k_v, G) \xrightarrow{\mu} \pi_1(G)_{\Gamma_k, tors}$$

is exact and functorial in G.

3.2. Let $L = (\overline{G}, \kappa)$ be a connected k-kernel. Then κ defines a k-form G^{tor} of $\overline{G}^{tor} = \overline{G}^{red}/\overline{G}^{ss}$. Let Z, Z^{ss} and Z^{sc} be as in 2.2. The short exact sequence of complexes

$$1 \to (Z^{sc} \xrightarrow{\rho} Z^{ss}) \to (Z^{sc} \to Z) \to (1 \to G^{tor}) \to 1$$

gives the exact sequence

$$\cdots \to H^3(k, \mathrm{Ker}\rho) \to H^2_{ab}(k, L) \to H^2(k, G^{tor}) \to \cdots$$

So, if k is a nonarchimedean local field, we have the inclusion $H^2_{ab}(k, L) \hookrightarrow H^2(k, G^{tor})$. Theorem 2.2 and Tate-Nakayama duality yields the following

Theorem 3.2.1. ([B5]) *Let $L = (\overline{G}, \kappa)$ be a connected k-kernel, where k is a nonarchimedean local field of characteristic zero. Assume that G^{tor} is k-anisotropic. Then, any element of $H^2(k, L)$ is neutral.*

Now suppose k is a number field. By [B1], $H^2_{ab}(k, L)$ is the fiber product of $H^2(k, G^{tor})$ and $\prod_{v\in\mathcal{V}_\infty} H^2_{ab}(k_v, L)$ over $\prod_{v\in\mathcal{V}_\infty} H^2(k_v, G^{tor})$. By Theorems 2.2, 3.2.1, we have the Hasse principle for H^2

Theorem 3.2.2. ([B5]) *Let $L = (\overline{G}, \kappa)$ be a connected k-kernel where k is a number field. Assume*

$$\mathrm{Ker}^2(k, G^{tor}) := \mathrm{Ker}(H^2(k, G^{tor}) \to \prod_{v\in\mathcal{V}} H^2(k_v, G^{tor})) = 0.$$

Then $\eta \in H^2(k, L)$ is neutral if and only if $loc_v(\eta) \in H^2(k_v, L)$ is neutral for all $v \in \mathcal{V}$.

4. Hasse principle and approximation theorems for homogeneous spaces. Cohomological obstructions

4.1. *Hasse principle.* Let X be a homogeneous space of a semisimple, simplyconnected group defined over a number field k. Fix $x \in X(\overline{k})$ and assume the stabilizer $\overline{H}$ of x is connected. For $\sigma \in \Gamma_k$, suppose $x^\sigma = xg_\sigma$, $g_\sigma \in G(\overline{k})$. Set

$$f(\sigma) = \mathrm{Int}(g_\sigma) \circ \sigma_* \quad \kappa = f \bmod \mathrm{Int}(\overline{H}).$$

Thus we get a k-kernel $L = (\overline{H}, \kappa)$. Let H^{tor} be a k-form of $\overline{H}^{tor}$ defined by κ. After Springer [Sp], consider the element $\eta(X) = Cl(f, u) \in H^2(k, L)$ defined by $u_{\sigma,\tau} = g_{\sigma\tau} \cdot {}^{\sigma} g_{\tau}^{-1} \cdot g_{\sigma}^{-1}$ which is the obstruction to the existence of a principal homogeneous space over X. Namely, $\eta(X)$ is neutral if and only if there is a principal homogeneous space, defined over k, over X. On the other hand, we know a cohomological criterion 3.2.2 for $\eta(X)$ to be neutral. Thus we have

Theorem 4.1. ([B5]) *Notation being as above, assume* $\mathrm{Ker}^2(k, H^{tor}) = 0$. *Then the Hasse principal holds for* X.

Example 1. ([Ha2]) A projective homogeneous space of semisimple simply-connected k-group, i.e. $\overline{H}$ is a parabolic subgroup.

In fact, H^{tor} is a quasi-trivial torus.

Example 2. ([Ku]) A spherical affine homogeneous space of a semisimple, simplyconnected k-group.

For an indecomposable affine homogeneous space, $\dim H^{tor} \leq 1$ from the classification of such spaces ([Br],[Kr]).

4.2. *Weak approximation.* Let X be a right homogeneous space of a connected k-group G. Assume that X has a rational point x and the stabilizer H of x is connected. Let S be a nonempty finite subset of $\mathcal{V}$. Assume that G is semisimple and simplyconnected for simplicity. Then it is easy to see that $X(k)$ is dense in $X(k_S)$ if and only if $X(k_S) = X(k)G(k_S)$. Hence, we are led to study the image of the map

$$X(k)/G(k) \longrightarrow X(k_S)/G(k_S).$$

This is done using Theorem 3.1.3 and the following commutative diagram with exact columns:

$$\begin{array}{ccc} X(k)/G(k) & \to & X(k_S)/G(k_S) \\ \cap\downarrow & & \cap\downarrow \\ H^1(k,H) & \to & H^1(k_S,H) \\ \downarrow & & \downarrow \\ H^1(k,G) & \to & H^1(k_S,G) \end{array}$$

Using the notations in 3.1, we set

$$C'(H) = \text{the image of } \oplus_{v\in\mathcal{V}}\lambda_v, \quad C^S(H) = \text{the image of } \oplus_{v\notin S}\lambda_v$$

Then the obstruction is the group

$$C'(H)/C^S(H) \simeq \mathrm{Cok}(H^1(k, H^{tor}) \to H^1(k_S, H^{tor})) =: \mathrm{Cok}^1_S(k, H^{tor})$$

Theorem 4.2.([B3]) *Notations and assumptions being as above, X has the weak approximation propeerty with respect to S if and only if* $\mathrm{Cok}^1_S(k, H^{tor}) = 0$.

Example 1. Let K be a splitting field of H^{tor}. If any local extension K_w/k_v, $w|v \in S$, is cyclic, X has the weak approximation property with respect to S. In fact, $\mathrm{Cok}^1_S(k, H^{tor}) = 0$ by Tate-Nakayama duality.

Example 2. A projective homogeneous space of a semisimple, simplyconnected k-group.

Example 3. A spherical affine homogeneous space of semisimple, simply-connected k-group.

4.3. *Strong approximation.* Let X, G, H and S be as in 4.2. We assume that G has the strong approximation property with respect to S. By Kneser and Platonov, it is satisfied if and only if

i) $G^{red} = G^{sc}$

ii) For any simple k-factor G_i of G^{red}, $G_i(k_S)$ is non-compact.

Then it is easy to see that X has the strong approximation property with respect to S if and only if $X(\mathbf{A}^S) = X(k)G(\mathbf{A}^S)$. Assume further $S \subset \mathcal{V}_f$ and define $C_S(H)$ to be the image of $\oplus_{v \in S}\lambda_v$. The obstruction is

$$C'(H)/C_S(H) \simeq \mathrm{Cok}(H^1(k, H^{tor}) \to H^1(\mathbf{A}^S, H^{tor})) =: \mathrm{Cok}^{1,S}(H^{tor})$$

Theorem 4.3. *Notationa and assumptions being as above, X has the strong approximation property with respect to S if and only if* $\mathrm{Cok}^{1,S}(k, H^{tor}) = 0$.

Example 1. Let K be a splitting field of H^{tor}, finite Galois over k, and G_v denotes the decomposition group of a chosen $w|v \in \mathcal{V}$ in K/k. If for any $v \notin S$ with $G_v \neq 1$, there is $v' \in S$ such that $G_v = G_{v'}$, then X has the strong approximation property with respect to S. A particular case is that K/k is cyclic. A sufficient condition for a quadric to have the strong approximation property given in section 1 follows from this.

Example 2. A spherical homogeneous space of an absolutely almost simple, simplyconnected k-group has the strong approximation property with respect to S, $\#S \geq 1$..

5. Brauer-Manin obstruction

Let X be a smooth variety defined over a field k. The Brauer-Grothendieck group $Br(X)$ is defined to be the group of certain equivalence classes $[\mathcal{A}]$ of sheaves of $\mathcal{O}_X$-algebras, locally free as $\mathcal{O}_X$-modules such that $\mathcal{A} \otimes_{\mathcal{O}_X} \kappa(x)$ is a central simple algebra over the residue field $\kappa(x)$ of any $x \in X$ ([G]). So, we have the pairing

$$X(k) \times Br(X) \to Br(k) \; ; \quad (x, b) \mapsto b(x)$$

defined by $b(x) = [\mathcal{A} \otimes \kappa(x)]$ for $b = [\mathcal{A}]$.
Suppose k is a number field. We set

$$Br_a(X) = \mathrm{Ker}(Br(X) \to Br(X \otimes \bar{k}))/\mathrm{Im}(Br(k) \to Br(X))$$

For a finite subset S of $\mathcal{V}$, set

$$B^S(X) = \mathrm{Ker}(Br_a(X) \to \prod_{v \in S} Br_a(X \otimes k_v))$$

$$B_S(X) = \mathrm{Ker}(Br_a(X) \to \prod_{v \notin S} Br_a(X \otimes k_v))$$

$$B_\omega(X) = \cup_S B_S(X).$$

By local, global classfield theory, we have an exact sequence

$$(*) \qquad 0 \to Br(k) \to \oplus_{v\in\mathcal{V}} Br(k_v) \xrightarrow{\sum_v inv_v} \mathbf{Q}/\mathbf{Z} \to 0$$

Hasse principle: By (*), we have the following well-defined pairing

$$\prod_{v\in\mathcal{V}} X(k_v) \times B_\omega(X) \to ; \quad ((x_v), b) \mapsto \sum_v inv_v b(x_v)$$

which is zero on $X(k)\times B_\omega(X)$. Here we simply use $b(x_v)$ for $\bar{b}(x_v)$ for a representative $\bar{b} \in Br_a(X)$ of b. Let $(\prod_v X(k_v))^B$ denote the set of $(x_v) \in \prod_v X(k_v)$ such that $\sum_v inv_v b(x_v) = 0$ for all $b \in B_\phi(X)$.

We say that *the Brauer-Manin obstruction to the Hasse principle for X is unique* if $(\prod_v X(k_v))^B \neq \phi$ implies $X(k) \neq \phi$.

Weak approximation with respect to S: Suppose there is $x \in X(k)$. Then we have the following continuous pairing which does not depend on the choice of x;

$$X(k_S) \times B_S(X) \to \mathbf{Q}/\mathbf{Z} \; ; \; ((x_v), b) \mapsto \sum_{v\in S}(inv_v(b(x_v)) - inv_v(b(x)))$$

which is zero on $\overline{X(k)} \times B_S(X)$, where $\overline{X(k)}$ is the topological closure of $X(k)$ in $X(k_S)$. Let $X(k_S)^B$ be the left kernel.

We say that *the Brauer-Manin obstruction to the weak approximation with respect to S for X is unique if* $X(k_S)^B = \overline{X(k)}$.

Strong approximation with respect to S: Suppose there is $x \in X(k)$. Then we have the following continuous well-defined pairing (cf [Sa, 6.2])

$$X(\mathbf{A}^S) \times B^S(X) \to \mathbf{Q}/\mathbf{Z} \; ; \; ((x_v), b) \mapsto \sum_{v\notin S}(inv_v(b(x_v)) - inv_v(b(x)))$$

which is zero on $\overline{X(k)} \times B^S(X)$, where $\overline{X(k)}$ is the topological closure of $X(k)$ in $X(\mathbf{A}^S)$. Let $X(\mathbf{A}^S)^B$ denote the left kernel.

We say that *the Brauer-Manin obstruction to the strong approximation with respect to S for X is unique if* $X(\mathbf{A}^S)^B = \overline{X(k)}$.

Now let X be a homogeneous space of a connected affine k-group G and S is a non-empty subset of $\mathcal{V}$.

Theorem 5.1.([B6]) *Let $\bar{x} \in X(\bar{k})$ and assume the stabilizer of $\bar{x}$ is connected. Suppose $\prod_{v\in\mathcal{V}} X(k_v) \neq \phi$. Then, the Brrauer-Manin obstruction to the Hasse principle for X is unique.*

Theorem 5.2.([B6]) *Assume there is $x \in X(k)$ and the stabilizer of x is connected. Then, the Brauer-Manin obstruction to the weak approximation with respect to S for X is unique.*

Theorem 5.3. *Assume that there is $x \in X(k)$ and the stabilizer of x is connected. Assume further that $S \subset \mathcal{V}_f$ and the strong approximation property with respect to S for G holds. Then, the Brauer-Manin obstruction to the strong approximation with respect to S for X is unique.*

To prove Theorem 5.1, 5.2, note that we may assume G^{ssu} is simply-connected ([B6 5.1]), $G^{ssu} = \mathrm{Ker}(G \to G^{tor})$. Then we use the fibration $X \to X/G^{ssu}$ to reduce to the case of a homogeneous space of a torus or simplyconnected group. The latter cases were already treated by Voskresenskii, Sansuc and Borovoi. For more details, consult [B6]

Assume G is semisimple and simplyconnected. As Borovoi showed for the weak approximation in [B3], we can express the cohomological obstruction $\mathrm{Cok}^1_S(k, H^{tor})$ in terms of the Brauer group of X, owing to the isomorphism

$$\pi_1(H)_{\Gamma_k, tors} \simeq \mathrm{Pic}(H)^D = \text{ dual of } \mathrm{Pic}(H)$$

for a connected affine k-group H and relating the Picard group with Brauer group by Sansuc's sequences [Sa,6.c]. The similar argument also works for the strong approximation property under certain assumptions and yields Theorem 5.3.

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